Please check the examination details below before entering your candidate information

Candidate surname	Other names

Pearson Edexcel International Advanced Level

Centre Number	Candidate Number

Wednesday 21 October 2020

Afternoon (Time: 1 hour 30 minutes) | Paper Reference **WFM01/01**

Mathematics

International Advanced Subsidiary/Advanced Level
Further Pure Mathematics F1

You must have:
Mathematical Formulae and Statistical Tables (Blue), calculator

Total Marks

Candidates may use any calculator permitted by Pearson regulations. Calculators must not have the facility for symbolic algebra manipulation, differentiation and integration, or have retrievable mathematical formulae stored in them.

Instructions

- Use **black** ink or ball-point pen.
- If pencil is used for diagrams/sketches/graphs it must be dark (HB or B).
- **Fill in the boxes** at the top of this page with your name, centre number and candidate number.
- Answer **all** questions and ensure that your answers to parts of questions are clearly labelled.
- Answer the questions in the spaces provided – *there may be more space than you need.*
- You should show sufficient working to make your methods clear. Answers without working may not gain full credit.
- Inexact answers should be given to three significant figures unless otherwise stated.

Information

- A booklet 'Mathematical Formulae and Statistical Tables' is provided.
- There are 8 questions in this question paper. The total mark for this paper is 75.
- The marks for **each** question are shown in brackets – *use this as a guide as to how much time to spend on each question.*

Advice

- Read each question carefully before you start to answer it.
- Try to answer every question.
- Check your answers if you have time at the end.
- If you change your mind about an answer, cross it out and put your new answer and any working underneath.

Turn over ▶

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Pearson

1. $$\mathrm{f}(x) = x^3 - \frac{10\sqrt{x} - 4x}{x^2} \qquad x > 0$$

(a) Show that the equation $\mathrm{f}(x) = 0$ has a root α in the interval $[1.4, 1.5]$ **(2)**

(b) Determine $\mathrm{f}'(x)$. **(3)**

(c) Using $x_0 = 1.4$ as a first approximation to α, apply the Newton-Raphson procedure once to $\mathrm{f}(x)$ to calculate a second approximation to α, giving your answer to 3 decimal places. **(2)**

a. $f(1.4) = (1.4)^3 - \frac{10\sqrt{1.4} - 4(1.4)}{(1.4)^2} = -0.4356732481$ $\qquad f(1.4) < 0$

$f(1.5) = (1.5)^3 - \frac{10\sqrt{1.5} - 4(1.5)}{(1.5)^2} = 0.5983561271$ $\qquad f(1.5) > 0$

There is a sign change from negative to positive, therefore a root, α, lies between $x = 1.4$ and $x = 1.5$.

b. $f(x) = x^3 - \frac{10\sqrt{x} - 4x}{x^2}$

rewrite f(x) as indices:

$f(x) = x^3 - \frac{10x^{\frac{1}{2}} - 4x}{x^2}$

$f(x) = x^3 - \left(\frac{10x^{\frac{1}{2}}}{x^2} - \frac{4x^1}{x^2}\right)$

$f(x) = x^3 - 10x^{-\frac{3}{2}} + 4x^{-1}$

$\boxed{f'(x) = 3x^2 + 15x^{-\frac{5}{2}} - 4x^{-2}}$

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Question 1 continued

c. $X_1 = 1.4$

we are looking for X_2

Numerical Sol^ns of Eq^ns:

Newton-Rhapson iteration for solving $f(x)=0$

$$X_{n+1} = X_n - \frac{f(X_n)}{f'(X_n)}$$

From Newton-Rhapson iteration: $X_2 = X_1 - \frac{f(X_1)}{f'(X_1)}$

(1) must find $f(X_1)$ by subbing in $X_1(1.4)$ into $f(x)$

(2) must find $f'(X_1)$ by differentiating $f(x)$ and subbing in $X_1(1.4)$ into $f'(x)$.

(1) $f(1.4) = -0.4356732481$ use part (a)

(2) $f'(x) = 3x^2 + 15x^{-\frac{5}{2}} - 4x^{-2}$ from part (b)

$f'(1.4) = 3(1.4)^2 + 15(1.4)^{-\frac{5}{2}} - 4(1.4)^{-2} = 10.30720093$

$$X_2 = (1.4) - \frac{-0.4356732481}{10.30720093}$$

$X_2 = 1.44226823$

$\boxed{\alpha = 1.442 \ (3 s.f.)}$

Q1

(Total 7 marks)

Turn over

P 6 5 7 5 4 A 0 3 3 2

2. The quadratic equation

$$5x^2 - 2x + 3 = 0$$

has roots α and β.

Without solving the equation,

(a) write down the value of $(\alpha + \beta)$ and the value of $\alpha\beta$

(1)

(b) determine, giving each answer as a simplified fraction, the value of

(i) $\alpha^2 + \beta^2$

(ii) $\alpha^3 + \beta^3$

(4)

(c) determine a quadratic equation that has roots

$$(\alpha + \beta^2) \text{ and } (\beta + \alpha^2)$$

giving your answer in the form $px^2 + qx + r = 0$ where p, q and r are integers.

(4)

a. $x^2 - (\text{sum of roots})x + (\text{product of roots}) = 0$

let α, β be roots of quadratic

$x^2 - (\alpha + \beta)x + (\alpha\beta) = 0$

sum of roots : $\alpha + \beta$

product of roots: $\alpha\beta$

$5x^2 - 2x + 3 = 0$) $\div 5$

$x^2 - \frac{2}{5}x + \frac{3}{5} = 0$

i. $\alpha + \beta = \frac{2}{5}$

$\alpha\beta = \frac{3}{5}$

bi. $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$

$= \left(\frac{2}{5}\right)^2 - 2\left(\frac{3}{5}\right)$

$= -\frac{26}{25}$

$\boxed{\alpha^2 + \beta^2 = -\frac{26}{25}}$

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Question 2 continued

ii. $\alpha^3 + \beta^3 = (\alpha+\beta)^3 - 3\alpha\beta(\alpha+\beta)$

$= \left(\frac{2}{5}\right)^3 - 3\left(\frac{3}{5}\right)\left(\frac{2}{5}\right)$

$= -\frac{82}{125}$

$\boxed{\alpha^3 + \beta^3 = \frac{-82}{125}}$

c. $x^2 - (\text{sum of roots})x + (\text{product of roots}) = 0$

$x^2 - \left(\alpha + \beta^2 + \beta + \alpha^2\right)x + \left((\alpha+\beta^2)\times(\beta+\alpha^2)\right) = 0$

$x^2 - \left(\alpha^2+\beta^2 + \alpha+\beta\right)x + \left(\alpha\beta + \alpha^3 + \beta^3 + \alpha^2\beta^2\right) = 0$

$x^2 - \left(\alpha^2+\beta^2 + \alpha+\beta\right)x + \left(\alpha\beta + (\alpha\beta)^2 + \alpha^3+\beta^3\right) = 0$

use ans from part (bi)

use ans from part (bii)

$x^2 - \left(\left(-\frac{26}{25}\right) + \left(\frac{2}{5}\right)\right)x + \left(\left(\frac{3}{5}\right) + \left(\frac{3}{5}\right)^2 + \left(\frac{-82}{125}\right)\right) = 0$

$x^2 - \left(-\frac{16}{25}\right)x + \left(\frac{38}{125}\right) = 0$

$x^2 + \frac{16}{25}x + \frac{38}{125} = 0$

$x^2 + \frac{16}{25}x + \frac{38}{125} = 0$ $\quad \times 125$

$125x^2 + 80x + 38 = 0 \longleftarrow$ all integer coefficients

$\boxed{125x^2 + 80x + 38 = 0 \qquad p = 125,\ q = 80,\ r = 38}$

Turn over

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3. $$f(z) = z^4 + az^3 + bz^2 + cz + d$$

where a, b, c and d are integers.

The complex numbers $3 + i$ and $-1 - 2i$ are roots of the equation $f(z) = 0$

(a) Write down the other roots of this equation.

(2)

(b) Show all the roots of the equation $f(z) = 0$ on a single Argand diagram.

(2)

(c) Determine the values of a, b, c and d.

(5)

a. For a quartic eq^n with form: $ax^4 + bx^3 + cx^2 + dx + e$, there are 4 roots.

There are 3 possibilities: 4 real roots

2 real roots and 2 complex roots (conjugate pair)

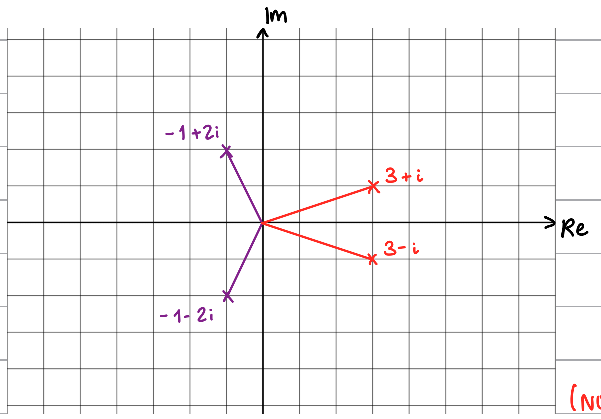
4 complex roots (2 conjugate pairs).

The other complex roots must be the complex conjugates of $(3+i)$ and $(-1-2i)$

(flip sign of imaginary component

$\boxed{3-i, -1+2i}$

b.



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Question 3 continued

c. $(z-(-1-2i))(z-(-1+2i))(z-(3+i))(z-(3-i))$

$(z+1+2i)(z+1-2i)(z-3-i)(z-3+i)$

$(z^2+z-\cancel{2iz}+z+1-\cancel{2i}+\cancel{2iz}+\cancel{2i}-4i^2)(z^2-3z+\cancel{iz}-3z+9-\cancel{3i}-\cancel{iz}+\cancel{3i}-i^2)$

$(z^2+2z+1-4(-1))(z^2-6z+9-(-1))$

$(z^2+2z+5)(z^2-6z+10)$

$(z^4-6z^3+10z^2-2z^3-12z^2+20z+5z^2-30z+50)$

$(z^4-8z^3+3z^2-10z+50)$

$z^4-8z^3+3z^2-10z+50$

$\boxed{a=-4,\ b=3,\ c=-10,\ a=50}$

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4. (a) Use the standard results for $\sum_{r=1}^{n} r^2$ and $\sum_{r=1}^{n} r$ to show that

$$\sum_{r=1}^{n} (2r-1)^2 = \frac{1}{3}n(4n^2-1)$$

for all positive integers n.

(5)

(b) Hence find the exact value of the sum of the squares of the odd numbers between 200 and 500

(4)

a. $\sum_{r=1}^{n} (2r-1)^2 = \sum_{r=1}^{n} 4r^2 - 4r + 1$

can split summation into 3

$\sum_{r=1}^{n} 4r^2 - \sum_{r=1}^{n} 4r + \sum_{r=1}^{n} 1$

factor 4 out summation

$= 4\sum_{r=1}^{n} r^2 - 4\sum_{r=1}^{n} r + \sum_{r=1}^{n} 1$

Summations

$\sum_{r=1}^{n} 1 = n$

$\sum_{r=1}^{n} r = \frac{1}{2}n(n+1)$

must learn!

$\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$

$\sum_{r=1}^{n} r^3 = \frac{1}{4}n^2(n+1)^2$

in the formulae booklet

$= 4\left[\frac{1}{6}n(n+1)(2n+1)\right] - 4\left[\frac{1}{2}n(n+1)\right] + [n]$

$= \frac{4}{6}n(n+1)(2n+1) - \frac{4}{2}n(n+1) + n$

$= \frac{2}{3}n(n+1)(2n+1) - 2n(n+1) + n$

factor out $\frac{1}{3}n$

$= \frac{1}{3}n\left[2(n+1)(2n+1) - 6(n+1) + 3\right]$

$= \frac{1}{3}n\left[4n^2 + \cancel{6n} + 2 - \cancel{6n} - 6 + 3\right]$

$= \boxed{\frac{1}{3}n\left[4n^2 - 1\right]}$ // shown

b. smallest odd no. from 200 - 500 : 201

$2r - 1 = 201$

$r = \frac{201+1}{2} = 101$ lower limit

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Question 4 continued

largest odd no. from 200 - 500 : 499

$2r - 1 = 499$

$r = \frac{499 + 1}{2} = 250$ upper limit

$\therefore$ new limits are $\sum_{r=101}^{250} (2r-1)^2$

$\sum_{r=101}^{250} (2r-1)^2 = \sum_{r=1}^{250} (2r-1)^2 - \sum_{r=1}^{100} (2r-1)^2$

$= \frac{1}{3}(250)\left(4(250)^2 - 1\right) - \frac{1}{3}(100)\left((100)^2 - 1\right)$

$= 20833250 - 1333300$

$= 19499950$

$\boxed{19499950}$

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5. The rectangular hyperbola H has equation $xy = 64$

The point $P\left(8p, \dfrac{8}{p}\right)$, where $p \neq 0$, lies on H.

(a) Use calculus to show that the normal to H at P has equation

$$p^3x - py = 8(p^4 - 1)$$

(5)

The normal to H at P meets H again at the point Q.

(b) Determine, in terms of p, the coordinates of Q, giving your answers in simplest form.

(4)

a. $xy = 64$

$y = \frac{64}{x} = 64x^{-1}$

$\frac{dy}{dx} = -64x^{-2}$

Sub in point P $(8p, \frac{8}{p})$

$\left.\frac{dy}{dx}\right|_{(8p, \frac{8}{p})} \quad -64(8p)^{-2} = \frac{-64}{(8p)^2} = \frac{-64}{64p^2} = \frac{\cancel{(64)}(-1)}{\cancel{(64)}(p^2)} = \frac{-1}{p^2}$

$M_{tangent} = \frac{-1}{p^2}$

$M_{normal} = p^2$

Set up line eqn $y - y_1 = m(x - x_1)$ with $m = p^2$

$(x_1, y_1) = (8p, \frac{8}{p})$

$y - \frac{8}{p} = p^2(x - 8p)$

$y - \frac{8}{p} = p^2x - 8p^3$

$p(y - \frac{8}{p}) = p(p^2x - 8p^3)$

$py - 8 = p^3x - 8p^4$

$8p^4 - 8 = p^3x - py$

$\boxed{8(p^4 - 1) = p^3x - py \text{ // shown}}$

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Question 5 continued

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b. (1) normal eqⁿ : $8(p^4-1) = p^3x - py$

(2) rectangular hyperbola : $xy = 64$

solve these 2 simultaneously to find points of intersection (coord. P & Q)

(2) $xy = 64$

$y = \frac{64}{x}$

sub into (1) and solve for x

$8(p^4-1) = p^3x - p\left(\frac{64}{x}\right)$

$\times x$

$x\left[8(p^4-1)\right] = x\left[p^3x - \frac{64p}{x}\right]$

$8x(p^4-1) = p^3x^2 - 64p$

$p^3x^2 - 8(p^4-1)x - 64p = 0$

$(p^3x + 8)(x - 8p) = 0$

$x = -8/p^3$ or $x = 8p$

$xy = 64$

$y = \frac{64}{x}$

$x = -\frac{8}{p^3}$, $y = \frac{64}{(-8/p^3)} = -8p^3$

$x = 8p$, $y = \frac{64}{(8p)} = \frac{8}{p}$

$\left(-\frac{8}{p^3}, -8p^3\right)$ or $\cancel{\left(8p, \frac{8}{p}\right)}$

↳ This is coord. P so reject

$\boxed{Q\left(-\frac{8}{p^3}, -8p^3\right)}$

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6. (i)

$$\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$$

(a) Describe fully the single transformation represented by the matrix **A**.
(2)

The matrix **B** represents a rotation of 45° clockwise about the origin.

(b) Write down the matrix **B**, giving each element of the matrix in exact form.
(1)

The transformation represented by matrix **A** followed by the transformation represented by matrix **B** is represented by the matrix **C**.

(c) Determine **C**.
(2)

(ii) The trapezium T has vertices at the points (–2, 0), (–2, k), (5, 8) and (5, 0), where k is a positive constant. Trapezium T is transformed onto the trapezium T' by the matrix

$$\begin{pmatrix} 5 & 1 \\ -2 & 3 \end{pmatrix}$$

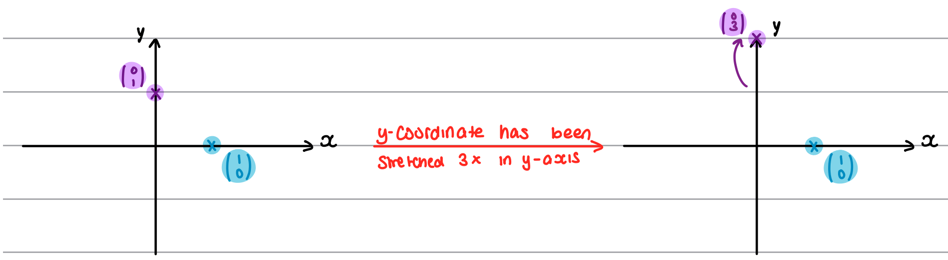
Given that the area of trapezium T' is 510 square units, calculate the exact value of k.
(5)

ia. Start by drawing a coordinate axis and mark coordinates $\begin{pmatrix}1\\0\end{pmatrix}$ and $\begin{pmatrix}0\\1\end{pmatrix}$

then mark the coordinates of new matrix

Identity matrix I $\qquad\qquad\qquad\qquad$ Matrix A

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \xrightarrow{\text{transformation}} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$$



$\boxed{\text{Stretch scale factor 3 parallel to the y-axis}}$

Question 6 continued

b.

$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ ← Anticlockwise rotation through θ about O (Given in Formulae booklet)

45° clockwise $= 315^\circ$ anticlockwise

sub in $\theta = 315^\circ \rightarrow \begin{bmatrix} \cos(315^\circ) & -\sin(315^\circ) \\ \sin(315^\circ) & \cos(315^\circ) \end{bmatrix} \rightarrow \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$

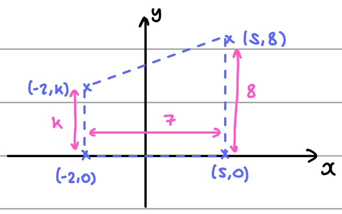
$\boxed{B = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}}$

c. order : $C = BA$

$C = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} (\frac{\sqrt{2}}{2})(1) + (\frac{\sqrt{2}}{2})(0) & (\frac{\sqrt{2}}{2})(0) + (\frac{\sqrt{2}}{2})(3) \\ (-\frac{\sqrt{2}}{2})(1) + (\frac{\sqrt{2}}{2})(0) & (-\frac{\sqrt{2}}{2})(0) + (\frac{\sqrt{2}}{2})(3) \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{3\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{3\sqrt{2}}{2} \end{bmatrix}$

$\boxed{C = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{3\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{3\sqrt{2}}{2} \end{bmatrix}}$

ii.



Area of Trapezium $= \frac{1}{2}(k+8) \times 7 = \frac{7(k+8)}{2}$ units^2

$|\det(M)| =$ area scale factor

$X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$\det(X) = (a)(d) - (b)(c)$

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Question 6 continued

$\det(M) = (5)(3) - (1)(-2)$

$= 17$

Area S.f. : $|\det(M)| = 17$

area of $T \times 17 =$ area of T'

$\frac{7(k+8)}{2} \times 17 = 510$

$\frac{7(k+8)}{2} = 30$

$7(k+8) = 60$

$k + 8 = \frac{60}{7}$

$k = \frac{60}{7} - 8 = \frac{4}{7}$

$\boxed{k = 4/7}$

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7. The parabola C has equation $y^2 = 4ax$, where a is a positive constant.

The line l with equation $3x - 4y + 48 = 0$ is a tangent to C at the point P.

(a) Show that $a = 9$

(4)

(b) Hence determine the coordinates of P.

(2)

Given that the point S is the focus of C and that the line l crosses the directrix of C at the point A,

(c) determine the exact area of triangle PSA.

(4)

a. $y^2 = 4ax$

$\frac{y^2}{4a} = x$

Sub in $x = \frac{y^2}{4a}$ into $3x - 4y + 48 = 0$

$3\left(\frac{y^2}{4a}\right) - 4y + 48 = 0$

$\frac{3y^2}{4a} - 4y + 48 = 0$

$3y^2 - 16ay + 192a = 0$

if line l is a tangent to C @ point P, then discriminant = 0 $(b^2 - 4ac = 0)$

discriminant: $(-16a)^2 - 4(3)(192a) = 0$

$256a^2 - 2304a = 0$

$a(256a - 2304) = 0$

$a = 0$ or $a = \frac{2304}{256} = 9$

However, Q states a is a positive constant $\therefore a = 9$ only.

$\boxed{a = 9}$

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Question 7 continued

b. To find point P, find intersection point between line and parabola by solving simultaneously.

(1) $y^2 = 4(9)x$

$y^2 = 36x$

(2) $3x - 4y + 48 = 0$

$3x - 4y + 48 = 0$

$3x = 4y - 48$

$36x = 48y - 576$

$(y^2) = 48y - 576$

$y^2 - 48y + 576 = 0$

$(y-24)^2 = 0$

$y = 24$

$(24)^2 = 36x$

$\frac{(24)^2}{36} = x = 16$

$\boxed{\therefore \text{Point } P(16, 24)}$

c. $a = 9$

$\therefore$ focus $S(9, 0)$

directrix eq^n $x = -9$

(1) Solve line ℓ and directrix eq^n simultaneously.

(1) $x = -9$

(2) $3x - 4y + 48 = 0$

Sub $x = -9$ into (2)

$3(-9) - 4y + 48 = 0$

Conics

	Ellipse	Parabola	Hyperbola	Rectangular Hyperbola
Standard Form	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$	$y^2 = 4ax$	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$	$xy = c^2$
Parametric Form	$(a\cos\theta, b\sin\theta)$	$(at^2, 2at)$	$(a\sec\theta, b\tan\theta)$ $(\pm a\cosh\theta, b\sinh\theta)$	$\left(ct, \frac{c}{t}\right)$
Eccentricity	$e < 1$ $b^2 = a^2(1-e^2)$	$e = 1$	$e > 1$ $b^2 = a^2(e^2-1)$	$e = \sqrt{2}$
Foci	$(\pm ae, 0)$	$(a, 0)$	$(\pm ae, 0)$	$(\pm\sqrt{2}c, \pm\sqrt{2}c)$
Directrices	$x = \pm\frac{a}{e}$	$x = -a$	$x = \pm\frac{a}{e}$	$x + y = \pm\sqrt{2}c$
Asymptotes	none	none	$\frac{x}{a} = \pm\frac{y}{b}$	$x = 0, y = 0$

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Question 7 continued

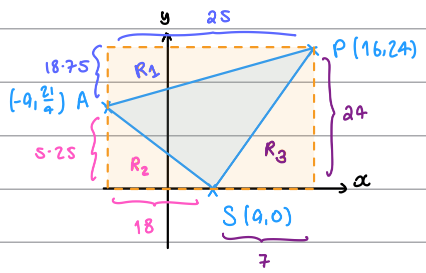
$-27 - 4y + 48 = 0$

$-4y + 21 = 0$

$4y = 21$

$y = \frac{21}{4}$

Point A $\left(-9, \frac{21}{4}\right)$



ΔPSA = Area of Rectangle $-\left(R_1 + R_2 + R_3\right)$

$= (24 \times 25) - \left(\left(\frac{1}{2} \times 18{\cdot}75 \times 25\right) + \left(\frac{1}{2} \times 5{\cdot}25 \times 18\right) + \left(\frac{1}{2} \times 7 \times 24\right)\right)$

$= \frac{1875}{8}$ units^2

$\boxed{\Delta PSA = \frac{1875}{8} \text{ units}^2}$

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8. (i) Prove by induction that, for $n \in \mathbb{Z}^+$

$$\sum_{r=1}^{n} \frac{2r^2 - 1}{r^2(r+1)^2} = \frac{n^2}{(n+1)^2}$$

(6)

(ii) Prove by induction that, for $n \in \mathbb{Z}^+$

$$f(n) = 12^n + 2 \times 5^{n-1}$$

is divisible by 7

(6)

i. (1) Base case: check true for $n = 1$

LHS: $\displaystyle\sum_{r=1}^{1} \frac{2r^2-1}{r^2(r+1)^2} = \frac{2(1)^2 - 1}{(1)^2(1+1)^2} = \frac{1}{4}$

RHS: $\displaystyle\frac{n^2}{(n+1)^2} = \frac{(1)^2}{(1+1)^2} = \frac{1}{4}$

LHS = RHS

∴ true for $n = 1$

(2) Assume true for $n = k$:

$$\sum_{r=1}^{k} \frac{2r^2-1}{r^2(r+1)^2} = \frac{k^2}{(k+1)^2}$$

(3) Inductive step, prove true for $n = k+1$:

$$\sum_{r=1}^{k+1} \frac{2r^2-1}{r^2(r+1)^2} = \sum_{r=1}^{k} \frac{2r^2-1}{r^2(r+1)^2} + \frac{2(k+1)^2 - 1}{(k+1)^2(k+1+1)^2}$$

← we are rewriting sum of (k+1) terms as the sum of (k) terms + $(k+1)^{th}$ term – we can now use step (2).

$$= \frac{k^2}{(k+1)^2} + \frac{2(k+1)^2 - 1}{(k+1)^2(k+2)^2}$$

Thinking ahead, sub in $n = k+1$ into RHS: this is what we are trying to prove using (2):

$$\sum_{r=1}^{k+1} \frac{2r^2-1}{r^2(r+1)^2} = \frac{(k+1)^2}{(k+1+1)^2}$$

Make a note so we know what we are working towards

$$= \frac{k^2(k+2)^2 + 2(k+1)^2 - 1}{(k+1)^2(k+2)^2}$$

$$= \frac{k^2(k^2+4k+4) + 2(k^2+2k+1) - 1}{(k+1)^2(k+2)^2}$$

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Question 8 continued

$$= \frac{k^4 + 4k^3 + 4k^2 + 2k^2 + 4k + 2 - 1}{(k+1)^2 (k+2)^2}$$

$$= \frac{k^4 + 4k^3 + 6k^2 + 4k + 1}{(k+1)^2 (k+2)^2}$$

$$= \frac{(k+1)^{\cancel{4}\,2}}{\cancel{(k+1)^2} (k+2)^2}$$

$$= \frac{(k+1)^2}{(k+2)^2}$$

$$= \frac{(k+1)^2}{(k+1+1)^2}$$ $\therefore$ true for $n = k+1$

(4) Conclusion

If this result is true for $n = k$, then it is true for $n = k+1$. As the result has been shown to be true for $n = 1$, then the result is true for all n.

ii. (1) Base Case: check true for $n = 1$

$f(1) = 12^{(1)} + 2 \times 5^{(1)-1}$

$= 12 + 2 \times 5^0$

$= 12 + 2 \times 1$

$= 14$

$= 2(7)$ $\longleftarrow$ divisible by 7.

$\therefore$ true for $n = 1$

(2) Assume true for $n = k$:

$f(k) = 12^k + 2 \times 5^{k-1}$

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Question 8 continued

(3) Inductive step, prove true for $n=k+1$:

$f(k+1) = 12^{k+1} + 2 \times 5^{k+1-1}$

$= 12^{k+1} + 2 \times 5^{k}$

$f(k+1) - f(k) = (12^{k+1} + 2 \times 5^{k}) - (12^{k} + 2 \times 5^{k-1})$

$= [(12)^1(12)^k + 2 \times (5^{k-1})(5^1)] - (12^k + 2 \times 5^{k-1})$

$= [12(12)^k + 10(5^{k-1})] - [12^k + 2(5^{k-1})]$

$= 12(12)^k + 10(5^{k-1}) - (12)^k - 2(5^{k-1})$

$= 11(12^k) + 8(5^{k-1})$

$= 11(12^k) + 4(2 \times 5^{k-1})$

$= 7(12^k) + 4(12^k) + 4(2 \times 5^{k-1})$

$= 7(12^k) + 4\underbrace{[12^k + 2 \times 5^{k-1}]}_{f(k)}$

$f(k+1) - f(k) = 7(12^k) + 4f(k)$

$\therefore f(k+1) = 7(12^k) + 5f(k)$ $\quad \therefore$ true for $n=k+1$

(4) Conclusion

Hence result is true for $n=k+1$. As true for $n=1$ and have shown if true for $n=k$ then it is true for $n=k+1$, so it is true for all $n \in \mathbb{N}$ by induction.

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